Building a Robustness Index

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Design Philosophy

- **Optimal design**: minimum cost & adequate performance.
- **Reliability-oriented optimal design**: the structural performance is usually judged based on reliability, which must be kept above a certain threshold.
- **One-level optimization**: instead of considering the reliability estimate as a self-standing optimization problem, it is included in the cost-benefit analysis by using the Kuhn and Tucker conditions ($F_{KT}$).
- **Idea**: also robustness can be introduced in the optimization problem as a further requirement for an adequate structural performance.
System Objectives

- Cost minimization $\rightarrow (F_C, w_C)$
- Reliability requirement $\rightarrow (-F_R, w_R)$
- Robustness $\rightarrow (-F_I, w_I)$

where the weights $w_X$ represent the Lagrange multipliers and increase with the safety margin within which the corresponding requirement is fulfilled.
Reliability & Robustness Oriented Optimal Design

Minimize:

\[ F(x, u_1, \ldots, u_m) = w_C F_C(x) + w_R F_R(x) + w_K F_K(x, u_1, \ldots, u_m) + w_I F_I(x) \]

where:

- \( x \) is the design parameters vector;
- \( u_i \) is the \( N \times 1 \) vector of the transformed random variables from the original space to the standard normal space, \( i = 1, \ldots, m \);
- \( m \) is the number of failure modes.

Reliability Estimate (Kuhn and Tucker Conditions)

Cost - (Reliability Requirement)

- (Robustness Requirement)
Objective 1: Cost

For a public structure whose reconstruction is systematic upon failure, which may only occur at the completion of the structure

\[
F_C(x) = C(x) + [C(x) + K] \left( \sum_{i=1}^{m} P_{fi}(x) \right) / \left( 1 - \sum_{i=1}^{m} P_{fi}(x) \right)
\]

where:

- \( C(x) \) is the design and construction cost of the structure;
- \( K = K_M + K_H \) is the sum of the direct cost of the structural failure (including both the damage and the debris removal), \( K_M \), and the cost of saving human lives, \( K_H \);
- \( P_{fi} \) is the probability of the \( i \)-th failure mode.
Objective 2a: Reliability Requirement

- Imposing: $P_{fi} \leq 10^{-6}$

  for $i=1,\ldots,m$ (number of failure modes)

- Is equivalent to minimize:

  $$F_R(x) = -\sum_{i=1}^{m} \frac{10^{-6} - P_{fi}(x)}{P_{fi}(x)}$$
Objective 2b: Reliability Estimate

- Kuhn and Tucker Conditions:

\[
F_{KT}(x, u_1, \ldots, u_m) = \sum_{i=1}^{m} \left| g_i(u_i, x) \right| + \sum_{j=1}^{N-1} \left( u_{ij} \left\| \nabla g_i(x) \right\| + \nabla g_{ij} \left\| u_i(x) \right\| \right)
\]

- where: \( g_i(u_i, x) \) is the \( i \)-th limit state function, with \( i=1, \ldots, m \).

- When, for any \( i \), both the contributes to \( F_{KT} \) are null, \( \|u_i\| \) represents the value of the reliability index \( \beta_i \) for the \( i \)-th failure mode, from which one determines \( P_{fi} \) as \( \Phi(-\beta_i) = \Phi(-\mu_{gi} / \sigma_{gi}) \), being \( \Phi(.) \) the standard normal cumulative distribution function.
Objective 3: Robustness

- Robust structures are those that develop the less catastrophic failure modes first → failure modes hierarchy

- Example: “weak beam/strong column” requirement in the design of a building
Portal-Frame
Numerical Example

$N = 9$ Random Variables

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Distribution</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1, M_3$</td>
<td>Normal</td>
<td>$\mu$ [kN m]</td>
<td>13.5 [kN m]</td>
</tr>
<tr>
<td>$M_3$</td>
<td>Normal</td>
<td>$\mu$ [kN m]</td>
<td>13.5 [kN m]</td>
</tr>
<tr>
<td>$M_{2c}, M_{4c}$</td>
<td>Normal</td>
<td>$\mu$ [kN m]</td>
<td>13.5 [kN m]</td>
</tr>
<tr>
<td>$M_{2b}, M_{4b}$</td>
<td>Normal</td>
<td>$\mu$ [kN m]</td>
<td>13.5 [kN m]</td>
</tr>
<tr>
<td>$H$</td>
<td>Normal</td>
<td>50 [kN]</td>
<td>15 [kN]</td>
</tr>
<tr>
<td>$V$</td>
<td>Normal</td>
<td>40 [kN]</td>
<td>15 [kN]</td>
</tr>
</tbody>
</table>
Robustness Index
Formulation

- Robustness Function: \[ Z = M_{pc} - M_{pb} \geq 0 \]

- Corresponding Robustness Index:

\[
I = \frac{\mu Z}{\sigma Z} = \frac{\mu_c - \mu_b}{\sqrt{\sigma_c^2 + \sigma_b^2}} \geq 0
\]

- \((\mu_b, \sigma_b), (\mu_c, \sigma_c)\) = means and standard deviations of the plastic moments of the beams and of the columns, respectively.
Optimization Problem Particularized for the Portal-Frame Example

- \( \mathbf{x} = [\mu_c, \mu_b]^T \) is the vector which collects the design parameters, so that:
  \[
  F_I (\mathbf{x}) = -I = \frac{x_2 - x_1}{\sqrt{2} \cdot 13.5}
  \]

- \( C (\mathbf{x}) = 3\mu_c + 2\mu_b \) is the design and construction cost;

- \( K = K_M + K_H \), where: \( K_M = 10^5 \) is the direct cost of the structural failure, (including both the damage and the debris removal); \( K_H = 2.4 \times 10^6 \) is the cost of saving human lives,

- Finally, \( m = 3 \) is the number of failure modes and \( P_{fi} \) is the probability of the \( i \)-th failure mode, computed by considering the following nonlinear limit state functions.
Nonlinear Limit State Functions

- Sway-mode: 
  \[ g_1 = M_1 + \min(M_{2c}, M_{2b}) + \min(M_{4c}, M_{4b}) + M_5 - hH \]

- Beam-mode: 
  \[ g_2 = \min(M_{2c}, M_{2b}) + 2M_3 + \min(M_{4c}, M_{4b}) - hV \]

- Complete mechanism: 
  \[ g_3 = M_1 + 2M_3 + 2\min(M_{4c}, M_{4b}) + M_5 - hH - hV \]

*The corresponding gradients must be computed piecewise.*
Solution via Differential Evolution (DE) Algorithm

Advantages:

- Search driven by objective function itself, instead of its gradient
- Independency of the results accuracy from the initial guess
- Few input parameters
- Moderate computational effort with respect to traditional Genetic Algorithms
- Easily adaptable to the solution of different problems
DE Solution Algorithm

Randomly generated initial population of $NP$ vectors of the design parameters: $x \in S = \text{search domain}$

$x_i (i = 1, \ldots, NP)$ = a possible candidate to form the next generation

Mutant vector: $v_i = x_{r1} + \Gamma \cdot (x_{r2} - x_{r3})$

Cross-over: $w_{ij} = v_{ij}$ if $\text{rand}_j \leq CR$, or $w_{ij} = x_{ij}$ else

Selection: $\begin{cases} F(w_i) < F(x_i) \rightarrow w_i \\ \text{else} \rightarrow x_i \end{cases}$

$\varepsilon = 10^{-4}$
$N = 5000$

Distance current optimal point from previous one $> \varepsilon$ or $\text{Nr. Iterations} < I$

$NP = 160$
Search range: $(500, 1) \text{kNm}$

$\Gamma = 0.8$
$CR = 0.8$

End
## DE – Input Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Population Size, $NP$</td>
<td>160</td>
</tr>
<tr>
<td>Search Range</td>
<td></td>
</tr>
<tr>
<td>- Upper Bound</td>
<td>500 kN m</td>
</tr>
<tr>
<td>- Lower Bound</td>
<td>1 kN m</td>
</tr>
<tr>
<td>Mutation Amplitude, $\Gamma$</td>
<td>0.5</td>
</tr>
<tr>
<td>Crossover Constant, $CR$</td>
<td>0.8</td>
</tr>
<tr>
<td>Tolerance for Convergence, $\varepsilon$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Max Iterations Nr., $I$</td>
<td>5000</td>
</tr>
</tbody>
</table>
Objective Function
Weights Calibration

\[ F(x, u_1, \ldots, u_m) = w_C F_C(x) + w_R F_R(x) + w_K F_K(x, u_1, \ldots, u_m) + w_I F_I(x) \]

where the weights \( w_x \) represent the Lagrange multipliers and increase with the safety margin within which the corresponding requirement is fulfilled:

<table>
<thead>
<tr>
<th>( w_C )</th>
<th>( w_R )</th>
<th>( w_K )</th>
<th>( w_I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>6</td>
<td>1</td>
<td>70</td>
</tr>
</tbody>
</table>
Results: objectives evaluated at the optimal point

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mu_c$</th>
<th>$\mu_b$</th>
<th>$I$</th>
<th>Cost</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without robustness</td>
<td>166.50</td>
<td>154.57</td>
<td>0.63</td>
<td>808.66</td>
<td>4212</td>
</tr>
<tr>
<td>With robustness</td>
<td>168.64</td>
<td>153.80</td>
<td>0.78</td>
<td>813.52</td>
<td>3098</td>
</tr>
</tbody>
</table>
## Reliability Results

<table>
<thead>
<tr>
<th>Failure mode</th>
<th>$g_i \times 10^{-5}$</th>
<th>$\beta_i$</th>
<th>$P_{fi} \times 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without robustness</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>column</td>
<td>9.91</td>
<td>5.2190</td>
<td>0.0090</td>
</tr>
<tr>
<td>beam</td>
<td>23.84</td>
<td>5.3943</td>
<td>0.0034</td>
</tr>
<tr>
<td>complete</td>
<td>-10.73</td>
<td>4.5932</td>
<td>0.2183</td>
</tr>
<tr>
<td>With robustness</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>column</td>
<td>15.4</td>
<td>5.1002</td>
<td>0.0170</td>
</tr>
<tr>
<td>beam</td>
<td>-0.5</td>
<td>5.0656</td>
<td>0.0204</td>
</tr>
<tr>
<td>complete</td>
<td>-55.43</td>
<td>4.6544</td>
<td>0.1624</td>
</tr>
</tbody>
</table>
Conclusions

- The robustness of a structure can be increased by including it in the design optimization problem. Of course, this happens with an increase of the costs.

- The desired hierarchy of failure modes can be achieved by properly adjusting the weights (Lagrangian multipliers) of the different terms in the objective function.
Future Developments

- Define extreme events scenarios and compute the conditional probabilities for each failure mode.
- Consider random failure in time by introducing out-crossing rates and interest rates.
- Generalize the process to different types of structural systems.