UNIVERSITY OF CALGARY

Factors affecting a risk-based interpretation of robustness

Marc A. Maes

University of Calgary, Canada

Outline of presentation

- We examine a recent risk-based interpretation of robustness
- We develop criteria that reflect specific robustness objectives
- We identify factors affecting these robustness criteria including:
  - consequence tail heaviness
  - component dependencies
  - common causes affecting consequence aggregation
  - instability due to load-sharing
- Conclusions

Introduction

- Robustness refers to the manner in which a system ”responds” to changes in variables affecting system states (“**disturbances**”)
- Specifically, **a robust structural system** is considered to be:
  - ”a system that will not loose functionality at a rate disproportional to the cause of a change in the state variables” (JCSS, 2008)
  - a system that ”contains” consequences of failure in response to certain disturbances (various structural design standards)
- In JCSS (2008), a **risk-based interpretation of robustness** is introduced:
  - direct consequences (associated with the states of the system’s components)
  - indirect consequences (associated with the states of the system)
  - robustness is tied to the ratio of direct versus indirect risk

Key aspects of robustness

- A careful definition is needed of what constitutes the structural **system**
- System robustness relates to **specific system performance objectives** (SPO), and this affects the characterization of consequences
  - SPO can be broad, as in: system survival, post-disaster operational capacity, etc.
  - SPO can be narrow and geared towards concepts intrinsic to structural design, such as; maintaining sufficient redundancy, etc.
- All **disturbances** must be identified and taken into account
- Robustness must account for:
  - all uncertainties associated with system assumptions, system objectives, the occurrence of disturbances and/or hazards
  - all model uncertainties involved in the response, cause-effect and consequence analyses

Indicators for robustness

- measures that are **not risk-based** i.e. non-probabilistic robustness indicators such as:
  - indices relating component member capacity to overall system capacity
  - measures of redundancy such as reserve strength ratios for different types of hazards
  - measures of progressive collapse
  - mechanistic measures based on energy balances subsequent to a system disturbance
  - measures involving the extent, propagation or propagation rate of structural damage
- measures that are risk-based
  - involving the consideration of consequences, exposure, uncertainties, and probabilistic system effects

Here, we focus on the second group of indicators

Lind’s indicator and generalization

- A system’s damage tolerance DT (=1/vulnerability) is defined as:

$$\mathrm{DT} = \frac{\Pr(F_S \mid R_0, S)}{\Pr(F_S \mid R_d(S), S)}$$

- The index DT ranges between P_{F0} and 1
- Lind’s damage tolerance can be loosely interpreted as robustness but it does not explicitly account for the consequences of system failure
- Generalization for multi-component systems

$$\mathrm{I}_{\mathrm{MCS}} = \frac{\Pr(F_S \mid \boldsymbol{R}, S)}{\max_i \Pr(F_S \mid \boldsymbol{R}_{-i}, S)}$$

- The robustness index $\mathrm{I}_{\mathrm{MCS}}$ is similarly based on a comparison between an undamaged and a damaged state
- The robustness index $\mathrm{I}_{\mathrm{MCS}}$ suffers from the same limitations as Lind’s measure

The risk assessment framework in JCSS

- The vulnerability of the system is the risk of **direct** consequences to all n_{CON} components. The direct risk R_D:

$$R_D = \sum_{k=1}^{n_{EXP}} \sum_{\ell=1}^{n_{CSTA}} p(\boldsymbol{C}_\ell \mid X_k) c_D(\boldsymbol{C}_\ell) p(X_k)$$

- The risk R_{ID} due to **indirect** consequences is assessed through the expected value of the indirect consequences with respect to all possible exposures and states:

$$R_{ID} = \sum_{k=1}^{n_{EXP}} \sum_{\ell=1}^{n_{CSTA}} \sum_{m=1}^{n_{SSTA}} c_{ID}(S_m, c_D(\boldsymbol{C}_\ell)) p(S_m \mid \boldsymbol{C}_\ell, X_k) p(\boldsymbol{C}_\ell \mid X_k) p(X_k)$$

- The robustness of a system can be quantified using a robustness indicator I_R:

$$\mathrm{I}_{\mathrm{R}} = \frac{R_D}{R_{ID} + R_D}$$

Direct versus indirect consequences

- The definition of the system is of tremendous significance in the definition of exposure, vulnerability and robustness
- It may be difficult to distinguish between c_D and c_{ID}:
  - for systems without clearly identifiable components such as soils or coastal/marine infrastructure, or
  - for systems that loose functionality gradually due to complex design and component interaction
- To avoid this difficulty, consider the **total** consequences c_T associated with all hierarchical levels within the system:

$$c_T = c_D + c_{ID}$$

  - this does not require the need to distinguish between c_D and c_{ID}
  - while the **expected value** R_T of the total consequences c_T governs decision making and risk management…
  - **… it is the upper tail of c_T which influences robustness**

Consequence Aggregation

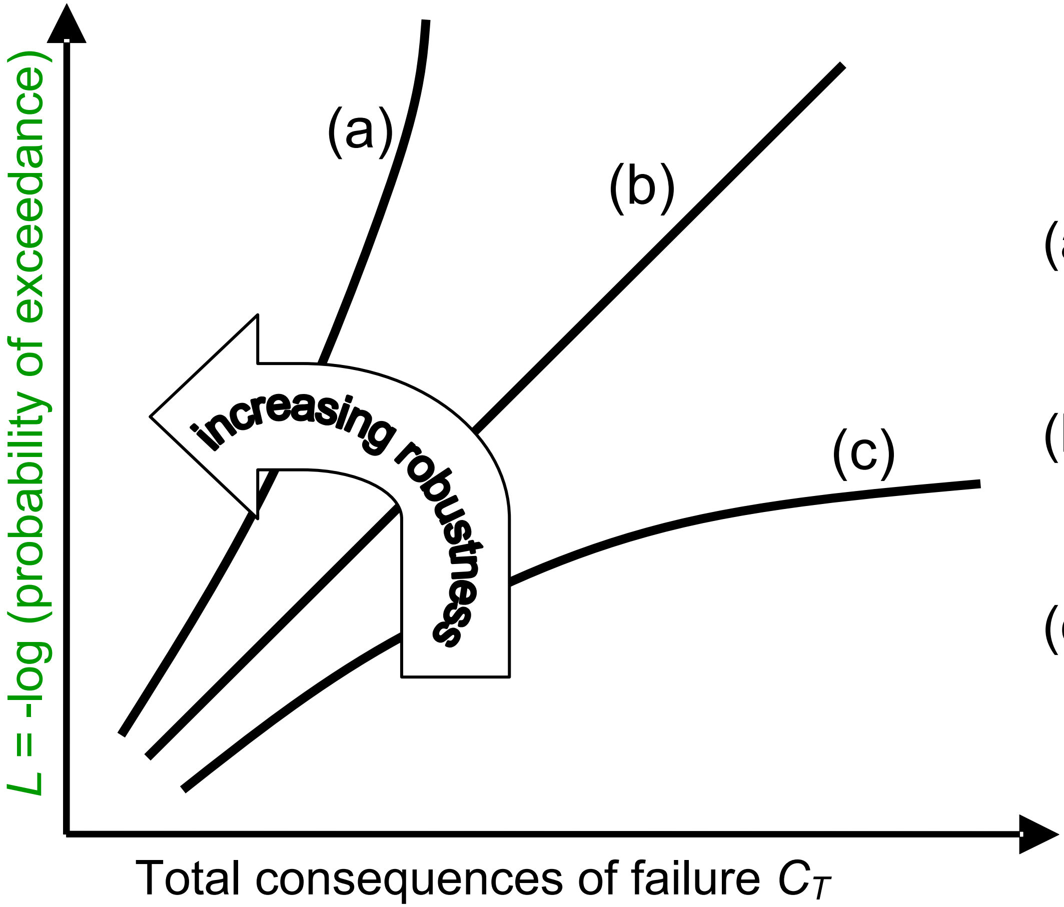
- Robustness can directly be assessed on the basis of the distribution of total consequences c_T because of the **aggregation process** required to derive $P(c_T)$

- Any disproportional response due to any disturbance can easily be spotted in the probability distribution of c_T:

  If a small disturbance Δy triggers a disproportionate shift or jump in the failure consequences, then this "instability" will, **through aggregation**, also show up in the cumulative distribution $F(c)$ of the total losses/consequences c in the form of a near zero slope which subsequently increases as a function of c.

- But since robustness critically focuses on the unexpected or disproportionate occurrence of larger consequences due to all possible small disturbances, it suffices to examine **the upper tail** of the total consequences.

L-plot of an upper tail segment



(a) fully contained consequences

(b) proportional (gradual) consequences

(c) out-of-control or blow-up consequences

The tail heaviness index *H*

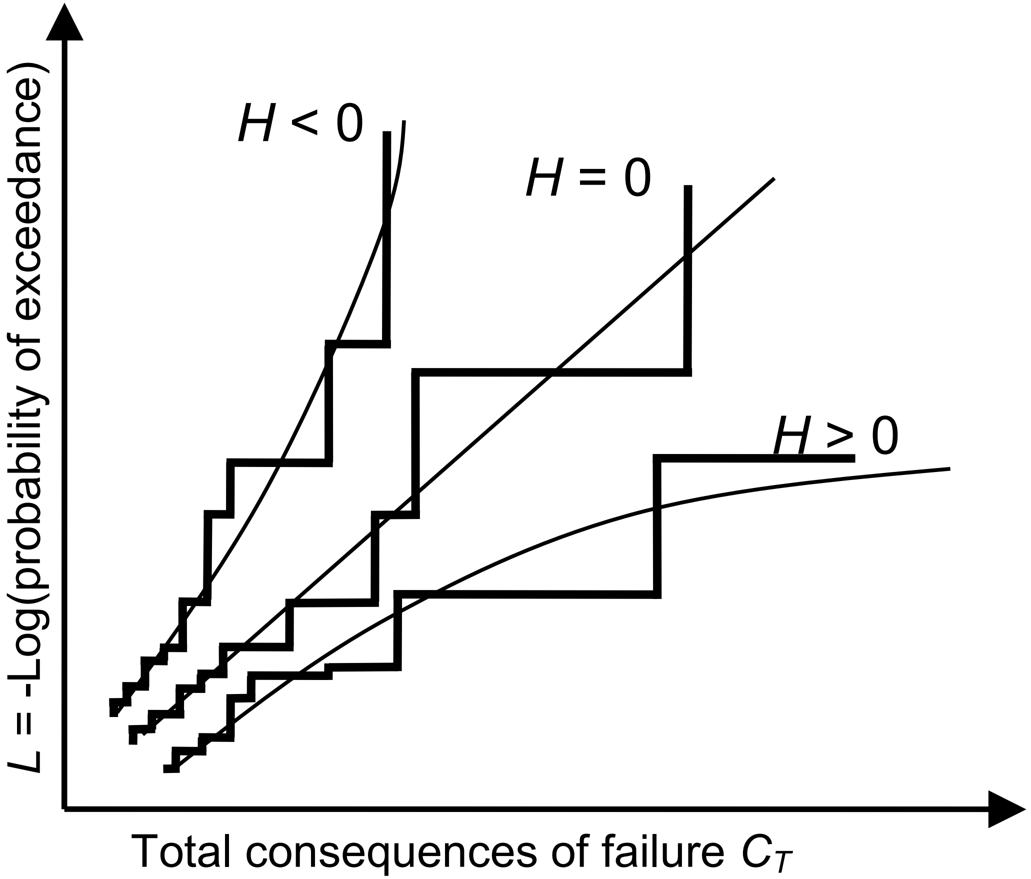
- The “containment of consequences” criterion can now be formulated in terms of the tail heaviness index *H*(*c*).

$$H(c) = -\left(\frac{L''}{L'^2}\right)_c = \left(-\frac{f'(1-F)}{f^2}\right)_c - 1$$

- *H*(*c*) can be calculated based on either:
  - the empirical distribution function of the total failure consequences *F*(*c*) or
  - using a smoothed *F*(*c*) or *L*(*c*)
  - it can be applied to the entire upper tail or any portion of it

- The tail heaviness index *H* is a powerful tool in statistical inference regarding high percentiles, tails and/or extreme values.

L-plot of empirical distributions



contain consequences *c* ⟺ *H* ≤ 0

Feller's ratio

As the level of consequences *c* becomes large, the ratio of the exceedance probabilities of the consequence levels *tc* and *c* should decrease to zero for a fixed number $t > 1$:

$$\frac{1-F_c(tc)}{1-F_c(c)} \Rightarrow 0 \qquad \text{as } c \to \infty$$

- it can be proved that this holds only for $L'' > 0$ or $H \leq 0$
- commonly used in the insurance industry

In large portfolio risk assessment, the reality (and the worry!) is that total losses are heavy tailed. When the ratio tends to a value $k \neq 0$ rather than 0, the marginal risk of large losses is in a run-away mode and, hence, not contained.

Equivalent requirements

To summarize the discussion, **the following robustness checks are equivalent**:

- aim to contain the total (aggregated) consequences in response to all possible disturbances
- suppress a disproportionate increase in aggregated consequence Δc at a high level of consequences $c(\boldsymbol{C}, S, x, y)$
- check that for critical c: $H(c) \leq 0$
- check that for large c, Feller's ratio decreases to 0

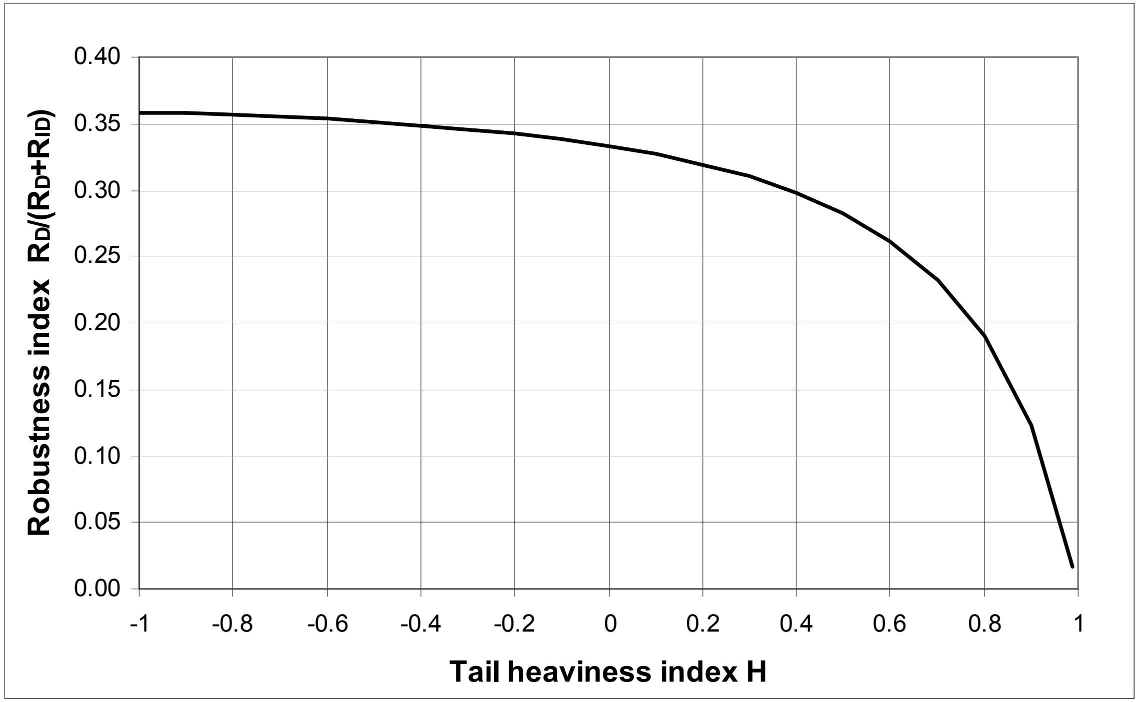
Insufficient robustness

In the context of consequence/risk analysis, lack of robustness can occur for the following reasons:

1. **heavy tail losses** e.g. due to indirect consequences
2. **dependencies** between components/elements in multi-component systems
3. **knowledge uncertainty** causing dependence in multi-component systems, or systems subject to multiple hazards
4. **load-sharing effects** causing dependent component failure in multi-component systems

1. Heavy tails

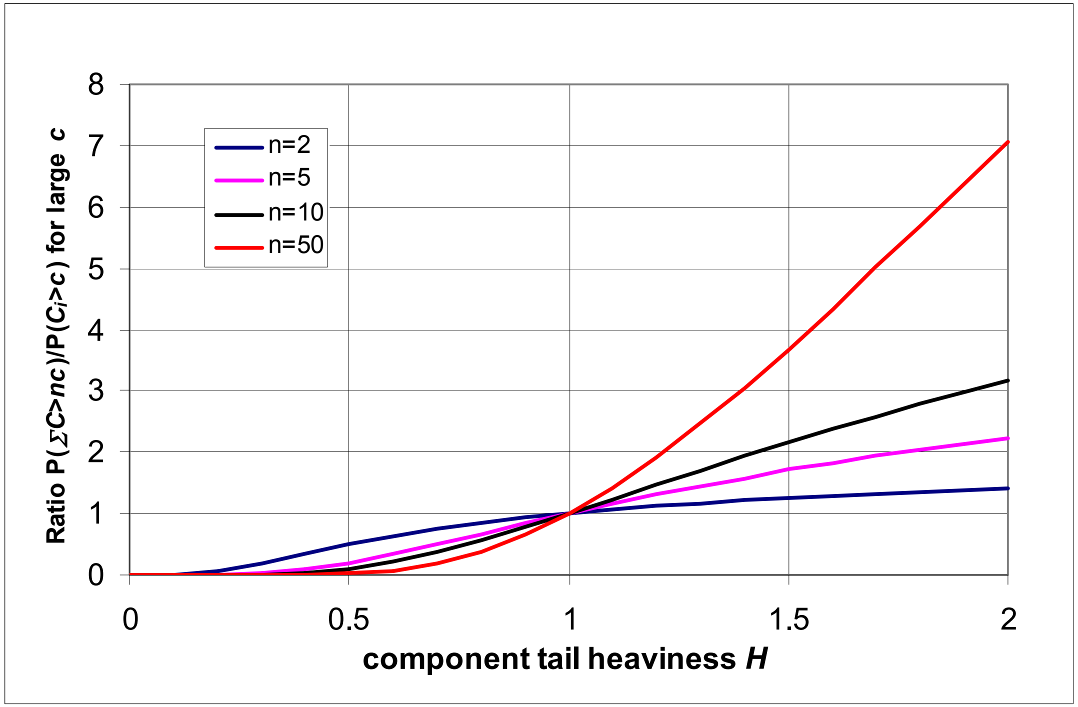
- stochastic branching: containment potential can easily be assessed using Feller’s criterion and H>0
- indirect consequences may lead to heavy tails
  example: c_I at 10^{-4} cumulative probability, but different H:



1. Heavy tails (ctd)

- *n* iid component losses each having heavy tails *H*. Use Feller's theorem to determine aggregate loss:

$$\frac{\Pr\left(\sum^{n} C_i > tc\right)}{\Pr(C_i > c)} \cong \frac{n}{t^{1/H}} \qquad \text{as} \quad c \to \infty \qquad \text{and } H > 0$$



2. Component dependencies

- System reliability is extremely sensitive to correlation between components (many references can be cited)

Example:

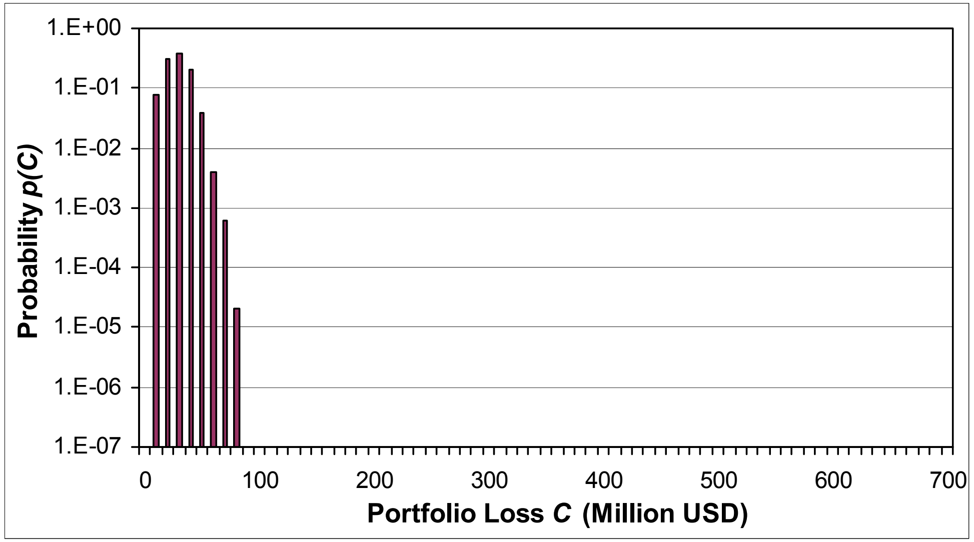
Consider a *k*-out-of-*n* system which does not lose functionality if at least *k* out of *n* constituents survive. If the failure probabilities of the *n* components share common uncertain variables *z* such as infrastructural variables/uncertainties, shared loads/hazards, or common environments, then the distribution of system failure consequences is given by:

$$f_c(c) = \int f_c(c \mid F_S, z) \sum_{i=n-k+1}^{n} \binom{n}{i} P(F_C \mid z)^i (1-P(F_C \mid z))^{n-i} f_Z(z)dz$$

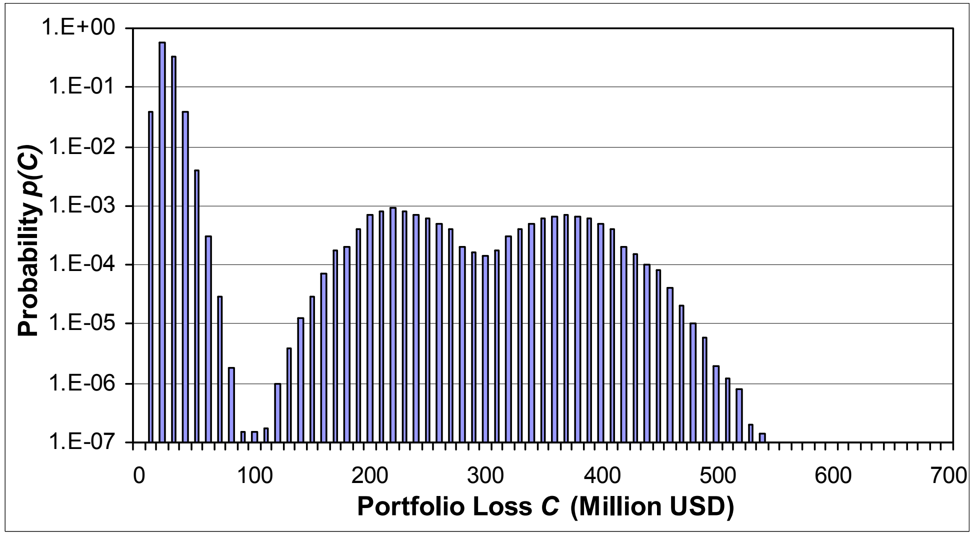
- Increasing the resulting correlation between components will increase $P(F_S)$ considerably, leading to a corresponding increase in tail consequences and decrease in robustness

3. Common knowledge uncertainties

- Often, model assumptions, model uncertainties and other epistemic uncertainties are shared among model components
- Example: Portfolio loss distribution (based on Bayraktarli and Faber, 2007)



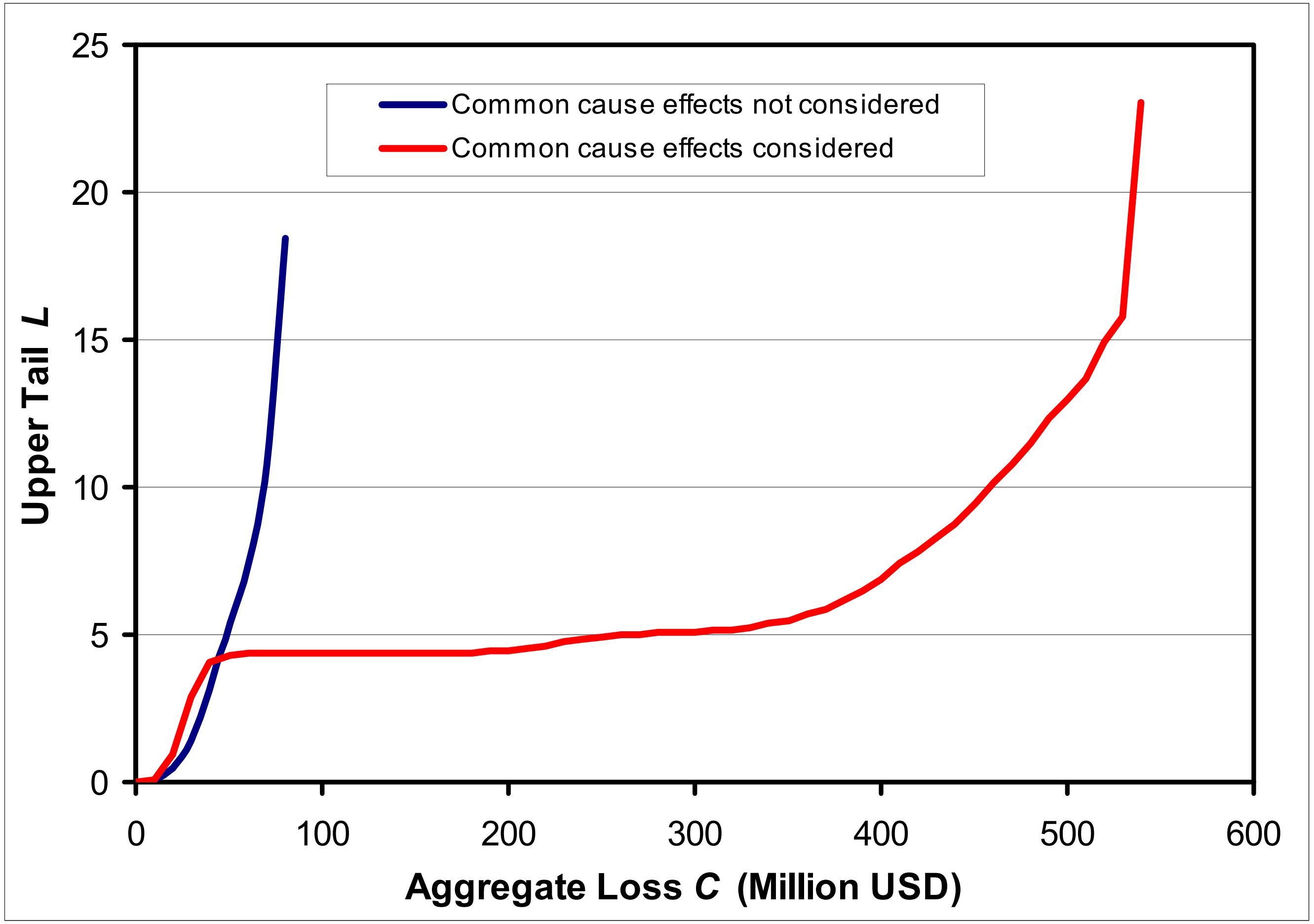
(a) not considering common cause effects



(b) considering common cause effects

3. Common knowledge uncertainties (ctd)

- *L*-plot of the portfolio losses clearly shows the *H*>0 segment



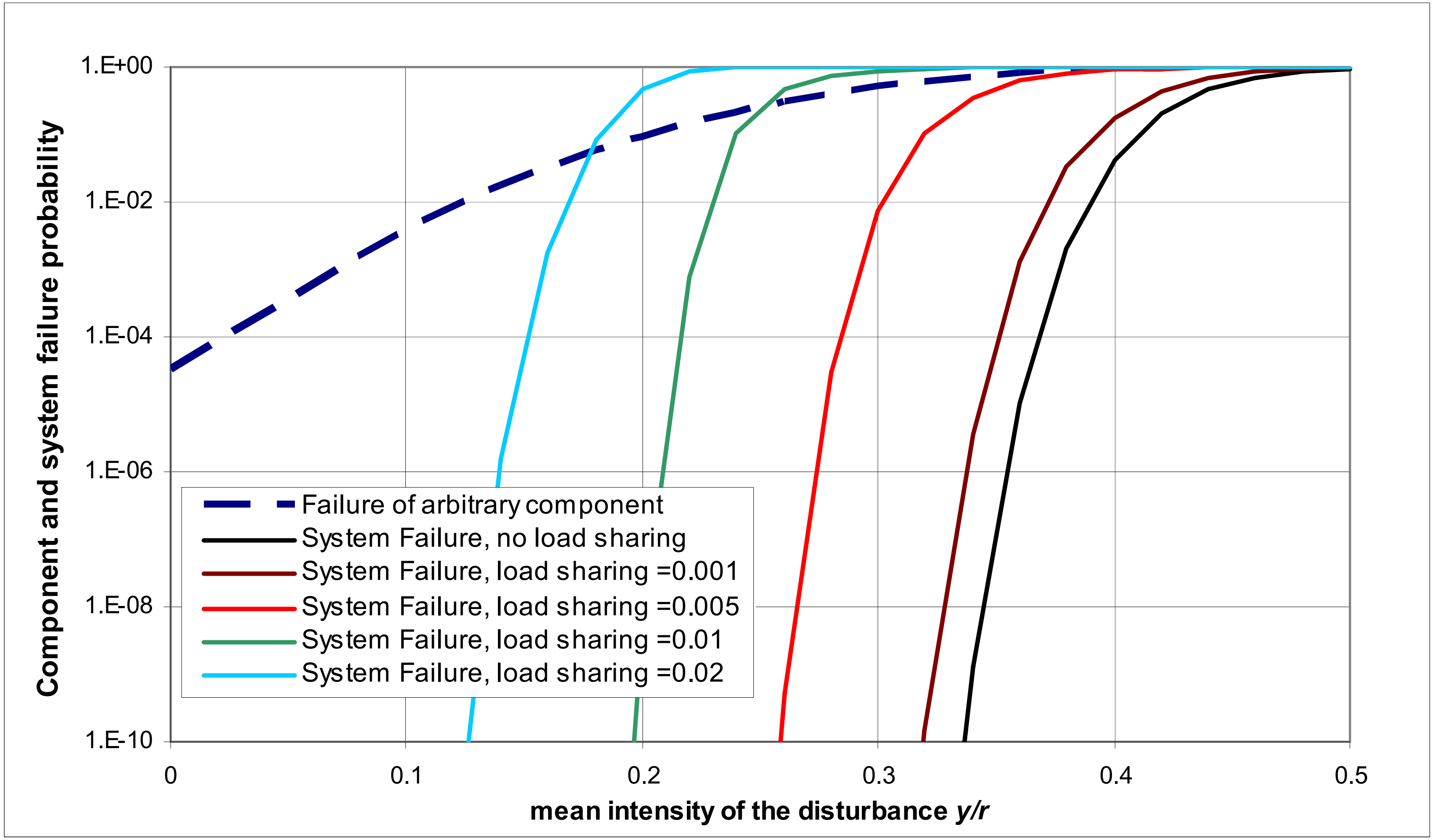
4. Load sharing effects

Dependence due to load sharing between components

- Following the failure of a component, the load may be re-routed and re-distributed to the remaining components. This is typical for e.g. electrical systems, power transmission
- Lack of robustness here is equivalent to cascading consequencies. Even the **smallest of load re-distributions** to the intact components can trigger a large increase in system failure risk
- Note that any geometric branching and progression of failure consequences, can easily be shown to result in a breakdown of Feller's condition, and hence, lack of robustness
- Example: load sharing in an *n*-component system:
  - all independent components are originally loaded at 70% of their (fixed) limiting capacity *r*
  - the system is subject to a disturbance which affects each component independently with mean $y \cdot r$ and a standard deviation of $0.05r$
  - failure of a component as a result of the disturbance, results in the load in each of the remaining components to be increased by a small amount $\Delta s/r$
  - system failure occurs when overload and failure occur in all *n* components

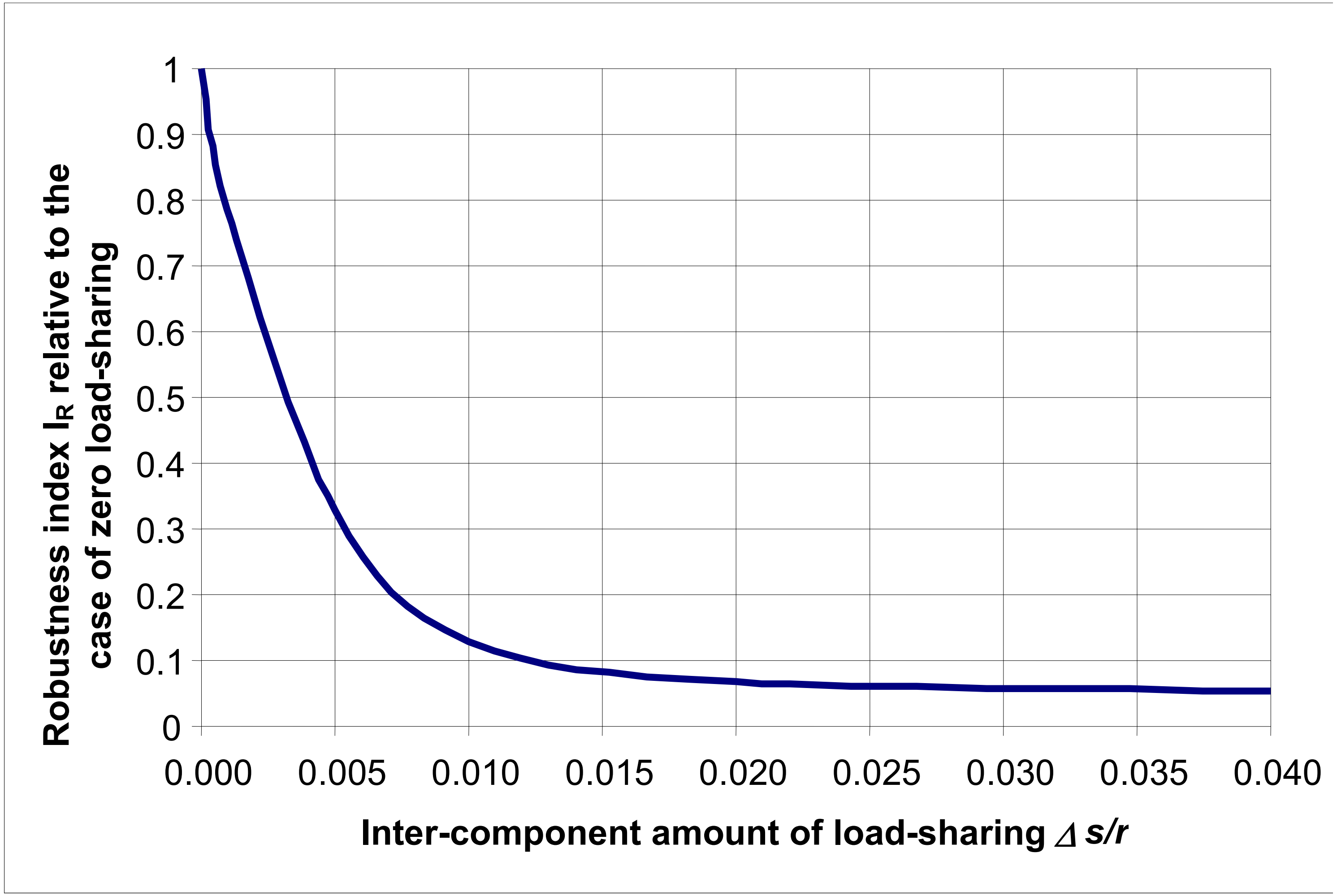
4. Load sharing effects (ctd)

System failure probability



4. Load sharing effects (ctd)

Resulting robustness index I_R



Conclusions

- The objective of containment and control of extreme consequences can be translated in a variety of tests or criteria related directly to the aggregated distribution of total consequences c_T
- The expected value of the total consequences governs decision making and the selection between alternatives; the upper tail distribution of c_T governs robustness (due to consequence aggregation)
- The statistical index H can easily be determined on the basis of the empirical distribution function $F(c)$ of total loss. It critically affects robustness: check $H>0$, or determine the Feller ratio for large c
- Inter-component dependencies reduce robustness
- Ignorance and model uncertainty reduce robustness
- Even slight load sharing following component failure reduces robustness by creating a potential for cascading types of failure